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On cavity cluster formation in a focused acoustic field

By K. A. MØRCH
Laboratory of Applied Physics, Technical University of Denmark, DK 2800, Lyngby, Denmark

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In the interior of an extended liquid a cyclic cavitation process can be set up by a focused acoustic wave field. A cluster of supercritical cavities is developed when the tensile strength of the liquid is exceeded. This occurs almost simultaneously in a confined region around the focal point where the sound speed rapidly vanishes when the cavities approach their critical size. As a consequence, the inner boundary condition for the acoustic field, carried by the single-phase liquid, changes from total reflection at the focal point without phase shift, as at a rigid wall, to reflection at the boundary of the cluster, which now forms a low-pressure two-phase kernel embedded in the single-phase liquid. The cluster is compliant to expansion waves but essentially rigid to compression waves. As soon as the cluster is formed its further development is determined jointly by the sound field and by the far field pressure of the liquid. The former first makes the cavities in the cluster grow and then contributes to its collapse, while the latter tends to bring about its collapse from the moment of cluster formation.

1. Introduction

In a focused acoustic field the pressure oscillations attain their maximum amplitude at the focal point, and at sufficient field intensity tensile stresses can be set up. These are sufficient to cause cavitation in an initially single-phase liquid. In general a cluster of small cavities is formed and collapses in each cycle (Ellis 1956; Plesset & Ellis 1955). It appears that the individual cavities in the cluster are nucleated simultaneously within a confined region, and at 20 kHz the inter-cavity distance is about 0.3 mm, and they then grow to a diameter typically of the order of 0.1 mm. The growth phase is followed by a cluster collapse phase in which the cavities collapse successively from the boundary of the cluster towards the cluster centre (Hansson & Mørch 1980; Mørch 1980, 1982, 1987*a*, *b*).

The cluster formation was previously discussed (Mørch 1986) assuming that near the focal point a two-phase quasi-equilibrium replaces wave propagation in the single-phase liquid, when cavities are nucleated in the liquid by rupture at the surface of a distribution of hydrophobic solid nuclei. The pressure at cavity nucleation was related to the size and form of these nuclei. A cavitation number for acoustic cavitation was found and the theory allows calculation of an upper limit of the cluster radius R_c. However, the calculated radii were about an order of magnitude larger than observed experimentally, which indicates that the nucleation criterion used is not the limiting one.

Thus, a reconsideration of the conditions of cavity nucleation is needed. In the present paper the theory is revised and cavity nucleation is defined in relation to the vanishing of the sonic speed in a liquid containing cavitation nuclei which approach their critical size. The definition is independent of whether the basic nuclei are gaseous or solid, but gas nuclei are chosen as the basis of calculation. The approach predicts simultaneous cavity nucleation within a region around the focal point, as observed in experiments, and explains the suppression of cavities outside this region.

In acoustically generated cavity clusters the inter-cavity distance is observed to be typically of the order of a few hundred μm, and consequently, a nuclei distribution of at least such a number density must be present during the cyclic cavitation process. It is pertinent to ask if these nuclei are the ones characterizing the tensile strength of the liquid in general. Greenspan & Tschiegg (1967) showed that the tensile strength of a liquid, defined by the stress at which the first cavitation event occurs in an acoustic field, is connected to the presence of solid nuclei. The tensile strength, which is about 1 bar in unfiltered water, increases if the maximum size of the solid nuclei in the water is reduced by filtration, until at nuclei diameters below 0.2 μm a tensile strength of about 200 bars is obtained. At the first cavitation event probably only a single or at most a few solid nuclei in the focal region of the sound field cause cavitation. However, it is likely that diffusion of gas into these cavities during their life-time leads to the formation of gaseous nuclei, and that their number is greatly increased in the following cavitation cycles owing to splitting of the gas content at collapse of the cavities. Further, these nuclei reduce the tensile strength of the liquid.

In experiments with a cavitation tube it was observed (I. Hansson, V. Kedrinskii & K. A. Mørch 1981, unpublished results) that the first stress pulse applied to tap water which has been at rest for about 2 days produces a tensile stress of about 1 bar while a successive pulse, if applied within a few hours, gives a tensile strength close to zero. The first stress pulse may be due to rupture at solid nuclei, which then leads to the formation of gas nuclei. It is expected, therefore, that a continuously repeated acoustic cavitation process leads to the formation of a cluster of gaseous nuclei in and around the focal region, and that here the tensile strength of the liquid is less than in the undisturbed liquid where the tensile strength is governed by solid nuclei.

In the following we assume that gas nuclei, all of the same size, are present inside as well as around the cluster region. However, it is likely that the nuclei shrink as the distance from the cluster increases, which of course strongly affects the conditions of cavitation nucleation in this zone.

2. The formation of the cavity cluster

The simplest form of a cavity cluster which can be generated and collapsed repeatedly is a spherical cluster produced by a converging spherical acoustic wave. Experimentally such a wave field was set up by Greenspan & Tschiegg (1967) and used for their investigations of the tensile strength of liquids by driving the sound field in a spherical bottle filled with water at its fundamental radial resonance frequency. Ellis (1956) studied an approximately similar cavitation process by high-speed photography using a cylindrical beaker filled with water and driven at cylindrical and axial resonance, so that a sound field comparable to a hemispherical wave was formed at the bottom of the beaker. Essentially hemispherical cavity clusters were generated and collapsed here. (A mirror cluster and wave field completes the analogy to the spherical case.)

To analyse the cluster formation we consider a single-phase liquid at equilibrium pressure p_0 in which a convergent, spherical acoustic wave field is propagated and totally reflected without phase shift at the focal point, so that a standing wave is generated.

The wave field has to satisfy the wave equation

$$\frac{\partial^2(r\phi)}{\partial t^2} = c_0^2 \frac{\partial^2(r\phi)}{\partial r^2}, \tag{1}$$

in which ϕ is the velocity potential and r and t are the radius and time, respectively, while c_0 is the sound speed of the liquid. It has solutions

$$r\phi = f(c_0 t - r) + g(c_0 t + r), \tag{2}$$

in which f represents the diverging and g the converging waves. For a converging wave of angular frequency $\omega = kc_0$,

$$g = C\cos(\omega t + kr), \tag{3}$$

we find the velocity perturbation

$$\Delta u_g = \left(-\frac{\partial \phi}{\partial r}\right)_g = \frac{kC}{r}\sin(\omega t + kr) + \frac{C}{r^2}\cos(\omega t + kr). \tag{4}$$

Here the value of C is related to the acoustic power W of the wave by

$$W = \left[\tfrac{1}{2}\rho_0\left(\frac{kC}{r}\right)^2 c_0 4\pi r^2\right]_{r\to\infty} = \frac{2\pi\omega^2}{c_0}\rho_0 C^2, \tag{5}$$

where ρ_0 is the density of the single-phase liquid.

By analogy we find for a diverging wave, resulting from total reflection of the converging wave at $r = 0$ without phase shift as at a rigid wall

$$f = -C\cos(\omega t - kr). \tag{6}$$

Its velocity perturbation

$$\Delta u_f = \left(-\frac{\partial \phi}{\partial r}\right)_f = \frac{Ck}{r}\sin(\omega t - kr) - \frac{C}{r^2}\cos(\omega t - kr), \tag{7}$$

and the two waves combine to give a standing wave

$$\Delta u = \Delta u_f + \Delta u_g = \frac{1}{r}\left(\frac{2W}{\pi\rho_0 c_0}\right)^{\frac{1}{2}}\left(\cos kr - \frac{\sin kr}{kr}\right)\sin\omega t, \tag{8}$$

which satisfies the inner boundary condition, that in a single-phase liquid $\Delta u \equiv 0$ for $r = 0$. The pressure perturbations are

$$\Delta p_g = \left(\rho\frac{\partial \phi}{\partial t}\right)_g = -\rho_0\frac{C}{r}\omega\sin(\omega t + kr), \tag{9}$$

and

$$\Delta p_f = \left(\rho\frac{\partial \phi}{\partial t}\right)_f = \rho_0\frac{C}{r}\omega\sin(\omega t - kr), \tag{10}$$

from which we obtain the pressure perturbation in the standing wave

$$\Delta p = \Delta p_f + \Delta p_g = -\frac{1}{r}\left(\frac{2\rho_0 c_0 W}{\pi}\right)^{\frac{1}{2}}\sin kr\cos\omega t. \tag{11}$$

The energy of the sound field continuously changes between kinetic energy and potential energy, the latter alternating between compressive and tensile stress.

A perfectly pure liquid can resist extremely large tensile stresses, but normal liquids contain gaseous and/or solid nuclei which give it a limited tensile strength, typically of the order of 0.1–1 bar. If the liquid is strained, cavities grow from the nuclei and pass a critical size at which the minimum equilibrium pressure in the liquid is obtained. This critical pressure p_{crit} is determined by the surface tension constant σ of the liquid and by the vapour and gas pressures p'_{v} and p'_{g} in the cavities and for solid nuclei also by their dimensions and surface properties. When the medium is strained beyond the critical condition the cavities grow, partly due to the straining itself, partly due to the relaxation of the tensile stress in the liquid phase. In this regime $\mathrm{d}p/\mathrm{d}\rho < 0$ for the medium, and a speed of propagation of low-frequency disturbances in the cavitating medium is not defined.

As a consequence, the above-mentioned field of standing waves can exist only if $p_0 + \Delta p > p_{\text{crit}} - p'_{\text{v}}$ during the complete cycle of the acoustic field. Actually, cavity growth changes the single-phase character of the liquid when the pressure has dropped to a cavity nucleation pressure p_{n} slightly higher than p_{crit}. This is because the nuclei yield significantly to the tensile stress applied, and as a consequence, the conversion of kinetic energy to potential energy (tensile stress) in the nucleation region is hampered, i.e. the inner boundary condition for the convergent wave changes. In figure 1 the pressure distribution around the focal point based on (11) is shown at different moments of time. The curve obtained for $\cos \omega t = -1$ gives the highest compressive stress in the focal region. The corresponding curve of tensile stress for $\cos \omega t = 1$ does not develop if cavitation nucleation occurs. Curve *a* shows the pressure distribution according to (11) when the nucleation pressure p_{n} is reached at the focal point, and the nuclei begin to influence the pressure field significantly. As time increases the locus of $p = p_{\text{n}}$ moves in a radial direction, first at infinite speed, then decreasing to sonic speed at a radius $r = R_{\text{n}} < R_{\text{max}}$, where R_{max} corresponds to $p = p_{\text{n}}$ for $\cos \omega t = 1$. Reflection of the converging wave at the boundary of the developing cavity cluster changes the pressure field and prevents the development of nucleation pressure for $r > R_{\text{n}}$. The influence of the cavitation nuclei is illustrated by curve *b* which shows that the pressure development in the focal region is restricted, and it indicates that the critical condition is obtained first at some distance from the focal point (or simultaneously throughout the focal region).

If we assume that the liquid contains gas nuclei of radius a their angular resonance frequency ω_{B} (Plesset & Prosperetti 1977) is

$$\omega_{\text{B}}^2 = 3\kappa \frac{p'_{\text{g}}}{\rho_0 a^2} - 2\frac{\sigma}{\rho_0 a^3}, \tag{12}$$

in which κ is the polytropic exponent of the gas and

$$p'_{\text{g}} = GT/a^3, \tag{13}$$

where G is a constant dependent on the gas content and T is the absolute temperature. For gas bubbles at equilibrium at the pressure p in the liquid (Knapp, Daily & Hammitt 1970),

$$p - p'_{\text{v}} = GT/a^3 - 2\sigma/a, \tag{14}$$

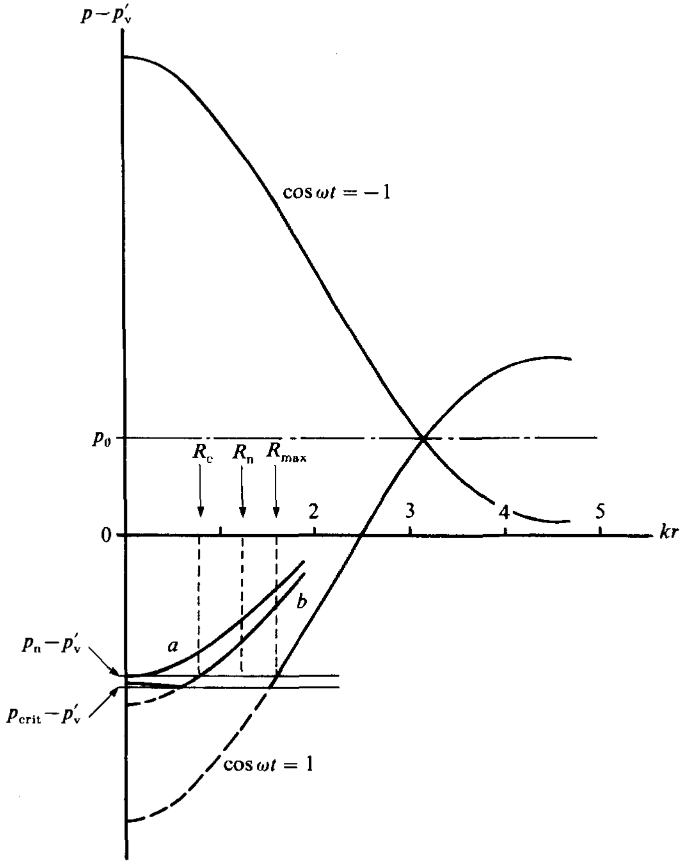

FIGURE 1. The maximum and minimum pressures (as obtained for $\cos\omega t = -1$ and $\cos\omega t = 1$) produced by a focused acoustic field *vs.* the distance r from the focal point. Curve a, pressure distribution at the limit of single-phase condition; curve b, pressure distribution at a time $t \approx t_n(R_c)$, after cavity nucleation and when critical conditions are approached in the central region.

from which the critical radius a_{crit} and pressure p_{crit} of the nuclei are found for $dp/da = 0$

$$a_{crit} = \left(\frac{3GT}{2\sigma}\right)^{\frac{1}{2}} = \frac{4\sigma}{3(p_v'-p_{crit})}, \tag{15}$$

and the corresponding gas pressure in the bubble is

$$p_{g,crit}' = \frac{2}{3}\frac{\sigma}{a_{crit}}. \tag{16}$$

For example, if in water at $T = 293$ K, where $\sigma = 0.0728$ N/m, the gas bubbles have a critical radius $a_{crit} = 3$ µm, then $p_{crit}-p_v' = -32.4$ kPa and $p_{g,crit}' = 16.2$ kPa. The radius shrinks to $a = 1.27$ µm at $p_0-p_v' = 10^5$ Pa with $p_g' = 2.14\times10^5$ Pa and from (12) we obtain $\omega_B = 18\times10^6\ s^{-1}$ for $\kappa = 1$, which applies for realistic ω, i.e. the bubble oscillations are isothermal (Plesset & Prosperetti 1977).

When the pressure p in the liquid is reduced ω_B is reduced, and $\omega_B = 0$ at $p = p_{crit}$, but ω_B remains high until very close to the critical condition. It turns out that at a frequency $f = \omega/2\pi = 20$ kHz of the acoustic field the gas nuclei can be considered in equilibrium at the prevailing pressure of the liquid essentially until the critical condition is reached.

The sound speed c in a liquid containing gas bubbles depends on the liquid itself as well as on the void fraction β of the gas bubbles. It is derived (Wijngaarden 1972) from

$$\rho = \rho_0(1-\beta)+\rho'\beta, \tag{17}$$

which by differentiation gives

$$\frac{d\rho}{dp} = c^{-2} = (1-\beta)\,c_0^{-2}+\beta c'^{-2}+(\rho'-\rho_0)\frac{d\beta}{dp}, \tag{18}$$

where ρ' and c' are the density and sound speed in the gas-vapour phase.

In general, the last term in (18) vanishes for $\beta \to 1$ and for $\beta \to 0$, and accordingly $c \to c'$ and $c \to c_0$, respectively, while at intermediate β-values this term becomes decisive, and a minimum sound speed is obtained for $\beta = 0.5$ (Wijngaarden 1980). However, at transition from gaseous to vaporous cavitation which occurs, in general, at small β-values, the above analysis is not sufficient.

If we assume that the nuclei are of uniform radius a, and their distribution can be considered an ordered structure of grid constant (inter-cavity distance) l then

$$\beta = q(a/l)^3, \tag{19}$$

in which e.g. an HCP-structure gives $q = 4\pi\sqrt{2}/3$.

It should be noted that such a structure is possible only for $a < \frac{1}{2}l$, above which the bubble structure must change to a drop structure.

With (15) we can eliminate p_v' and GT from (14) and the result can be written

$$p-p_{crit} = \tfrac{2}{3}\sigma a_{crit}^2(a^{-3}-a_{crit}^{-3})-2\sigma(a^{-1}-a_{crit}^{-1}), \tag{20}$$

from which

$$dp/da = 2\sigma(a^{-2}-a_{crit}^2/a^4). \tag{21}$$

From (18), (19) and (21) we obtain the dispersion equation, valid at frequencies low compared to the resonance frequency of the bubbles, and therefore, according to the above discussion, almost until $a = a_{crit}$,

$$\frac{1}{c^2} = \frac{1-\beta}{c_0^2}+\frac{\beta}{c'^2}+\frac{3\rho_0 a^3\beta}{2\sigma(a_{crit}^2-a^2)}, \tag{22}$$

in which $\rho' \ll \rho_0$ is used. It appears that even at low void fractions the influence of the gas bubbles becomes decisive and causes the sound speed c in the bubbly liquid to vanish when their radius approaches the critical value.

In figure 2 the cavity radius a and the sound speed c are shown *vs.* the pressure $p-p_v'$ for the case of a cluster of gas bubbles of critical radius $a_{crit} = 3$ µm as considered above, and at an inter-cavity distance $l = 300$ µm. The sound speed of the bubbly medium is little or only moderately influenced by the subcritical gas bubbles as long as the pressure is just a few kPa above the critical limit. At pressures close to the critical pressure the bubbles become decisive and the sound speed drops abruptly towards zero. Thus, at sufficient pressure reduction the acoustic impedance vanishes at the boundary of a developing cluster, where reflection of the convergent acoustic wave occurs as at a compliant wall, while only a vanishing part of the wave energy is transmitted into the cluster. In (1)–(11) the sound speed of the medium is assumed to be constant, but minor and gradual changes do not exclude their

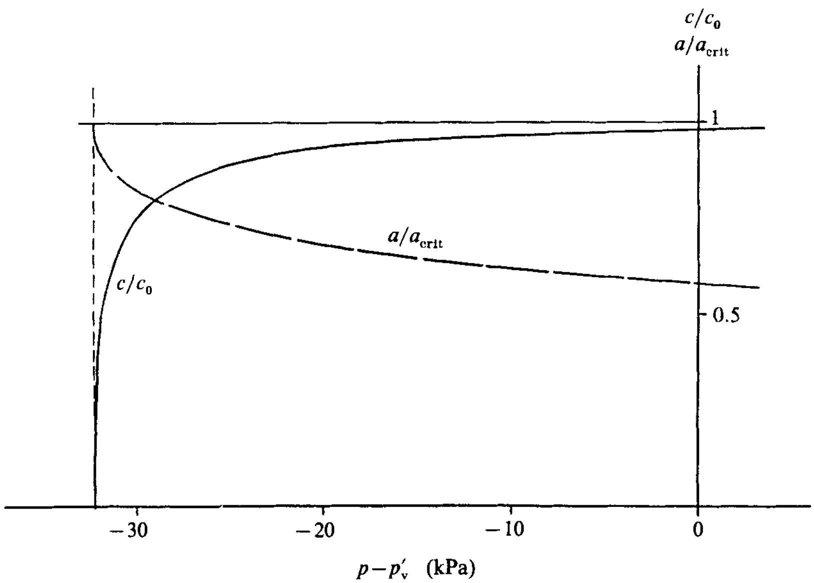

FIGURE 2. The cavity radius a and the sound speed c *vs*. pressure $p-p'_v$ for gas bubbles of critical radius 3 μm at an inter-cavity distance $l = 300$ μm in water of $\rho_0 = 998\ \mathrm{kg/m^3}$, $\sigma = 0.0728$ N/m and $c_0 = 1484$ m/s. $q = 4\pi\sqrt{2}/3$.

applicability. On the other hand, if a significant change of the local sound speed occurs within a distance of the order of the inter-cavity distance l the cluster can be considered to cause total reflection, and it can be dealt with as a cavitating medium. A boundary between minor and decisive influence of the cavities can suitably be defined by the cavity nucleation pressure p_n at which $c = \frac{1}{2}c_0$. This definition of p_n is an estimate. For the case considered in figure 2 where $p_{crit}-p'_v = -32.4$ kPa we obtain $p_n - p'_v = -31.8$ kPa. The disappearance of c therefore means $\Delta c/\Delta p = c_0/(2(p_n - p_{crit})) = 1.3$ m/(s·Pa) while, if $W = 50$ watts and the frequency $f = 20$ kHz, (11) gives $\mathrm{d}p/\mathrm{d}r \approx 1\times 10^6$ Pa/m, when the cluster radius obtained in the following paragraph is used. Accordingly, c vanishes over a distance of 600 μm $\approx 2l$.

In the following discussion we consider a medium containing subcritical gas bubbles to be a single-phase liquid for $p > p_n$ and a two-phase cavitating medium for $p_{crit} < p < p_n$. With the pressure perturbation (11) of the acoustic field superposed on the equilibrium pressure p_0 the single-phase condition (Mørch 1986) is therefore satisfied only if

$$p_0 + \Delta p > p_n - p'_v. \tag{23}$$

However, at sufficiently large acoustic power W the pressure drops below the limit of cavity nucleation during part of the cycle. The locus of nucleation then moves in a radial direction from $r = 0$ at $t = t_n(0)$. The relation between the time of cavity nucleation t_n and the radius r is obtained from (11) and (23) which give

$$\frac{kr}{\sin kr} = A\cos\omega t_n, \tag{24}$$

in which

$$A = \frac{\omega}{p_0 - p_n + p'_v}\left(\frac{2W\rho_0}{\pi c_0}\right)^{\frac{1}{2}} \tag{25}$$

For cavity nucleation to occur, we find from (24) that $A > 1$ is demanded. A is the cavitation number of acoustic cavitation. The locus of cavity nucleation moves in a radial direction at speed $v_n = dr/dt_n$ and by differentiating (24) we obtain

$$v_n = -Ac_0 \sin \omega t_n \sin kr/(1 - kr \cot kr). \tag{26}$$

It appears that $v_n \to \infty$ at values of $r \to 0$, and thus, in a region $r < R_n$ around the focal point $v_n > c_0$. Here, the wave field given by (11) may produce cavity nucleation, even though critical size cavities have developed already at a smaller radius. The relaxation waves they set up move outwards at only sonic speed in the liquid phase and do not catch up with the nucleation front until $v_n = c_0$. This is apparent from the r, t diagram in figure 3, where the whole domain I is governed by the standing wave pattern set up by total reflection without phase shift at $r = 0$.

The pressure changes in the liquid, resulting from the standing wave, reflect that the sound field exerts a force on the liquid elements. On the other hand, if the energy which is to be transformed from kinetic to potential energy by the sound field is larger than that which can actually be stored in the medium, the limit being given by the critical pressure p_{crit}, then the excess energy is not transformed and the boundary conditions for the converging wave change so that it is reflected with a phase shift. Due to the spherical symmetry, cavity nucleation ($p = p_n$) will occur simultaneously on a spherical surface which moves in a radial direction from the focal point. As wave energy is propagated only weakly beyond the nucleation zone and into this spherical volume because c vanishes here, the energy transformation resulting in a pressure drop below the nucleation pressure p_n takes place essentially within the volume by which the nucleation sphere grows in a small time interval considered, and therefore the cavities in the interior of the sphere grow only slowly.

If in a first approximation we disregard this slow growth of the inner cavities, then the rate of work done by the sound field on the spherical volume increases the tensile stress solely in the nucleation zone at its surface. The position where critical conditions are first reached determines the cluster radius because here the cavities reach unstable equilibrium and relaxation towards the vapour pressure occurs. This cluster radius R'_c is determined from the rate of work of the sound field per unit volume

$$-\Delta p \, \Delta u = v_n[(p_{crit} - p_0 - p'_v)^2 - (p_n - p_0 - p'_v)^2]/(2\rho_0 c_0^2),$$

in which Δu and Δp are given by (8) and (11). By use of (24)–(26) we obtain from the above equation

$$\delta p^* = \frac{p_n - p_{crit}}{p_0 - p_n - p'_v} = [1 + 2((kR'_c)^{-1} - \cot kR'_c)^2]^{\frac{1}{2}} - 1. \tag{27}$$

In figure 4, curve a shows (kR'_c) as a function of δp^*. The nucleation pressure p_n is always reached first at $r = 0$, but as the locus of nucleation initially moves at infinite speed (26), the nucleation zone moves outwards too fast to allow pressure reduction to the critical condition in this zone until a position $r = R'_c$ where v_n has dropped. Thus, R'_c is the location where a totally compliant boundary condition for the converging acoustic wave develops, while inside this sphere the cavities are still slightly subcritical. According to this model a spherical shell of vaporous cavities would be formed having the radius R'_c, and by relaxation the subcritical cavities in its interior would collapse.

Actually, as c has not completely vanished at pressures between the nucleation pressure and the critical pressure a small amount of wave energy is propagated beyond the nucleation zone, and it causes growth towards critical size of the cavities also inside the nucleation sphere, while the fast growth of those at its surface is

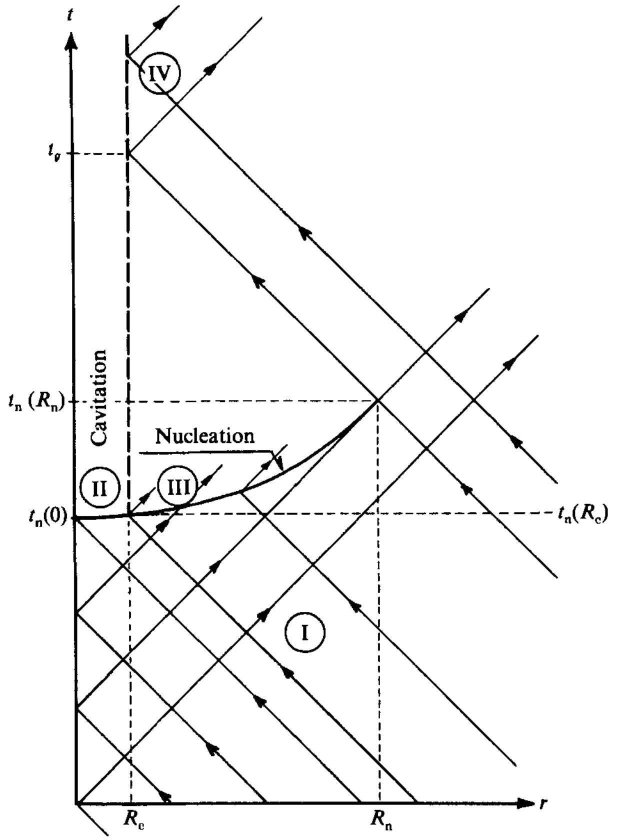

FIGURE 3. r,t diagram for the wave propagation in the focal region during cluster formation. Region I; total reflection at the focal point as at a rigid wall (equations (8) and (11)); region II: cavitation zone; region III: cavity nucleation zone; region IV: total reflection at cluster boundary as at a compliant wall (equations (30) and (31)).

slightly retarded. Therefore, it is to be expected that the critical condition occurs almost simultaneously in the whole domain, and that (27) underestimates the cluster radius. An alternative procedure would therefore be to calculate the radius R_c at which the work of setting up the stress field according to (11) corresponds to that demanded for reducing the pressure from p_0 to p_{crit} throughout the whole nucleation sphere, i.e.

$$\int_0^{R_c} (\Delta p^2)_{t_n(R_c)}\, 4\pi r^2\, dr = \tfrac{4}{3}\pi R_c^3 (p_{crit} - p_0 - p_v')^2,$$

which with (11) and (24) gives

$$\delta p^* = (\tfrac{3}{2}(1-\cos kR_c \sin kR_c/(kR_c))/\sin^2 kR_c)^{\frac{1}{2}} - 1. \tag{28}$$

Equation (28) is shown as curve b in figure 4 and is expected to give a more correct calculation of the cluster radius than curve a. This calculation fails, owing to its basis of integration, to give the information that the critical condition tends to develop first at some distance from the focal point – or, as a consequence of the weak wave propagation into the cavitation region, simultaneously throughout the cluster.

For the above example with gas bubbles of critical radius $a_{crit} = 3.0$ µm, $l = 300$ µm we find $\delta p^* \approx 4.5 \times 10^{-3}$ when $p_0 = 10^5$ Pa and, accordingly, from (28) $kR_c = 0.25$

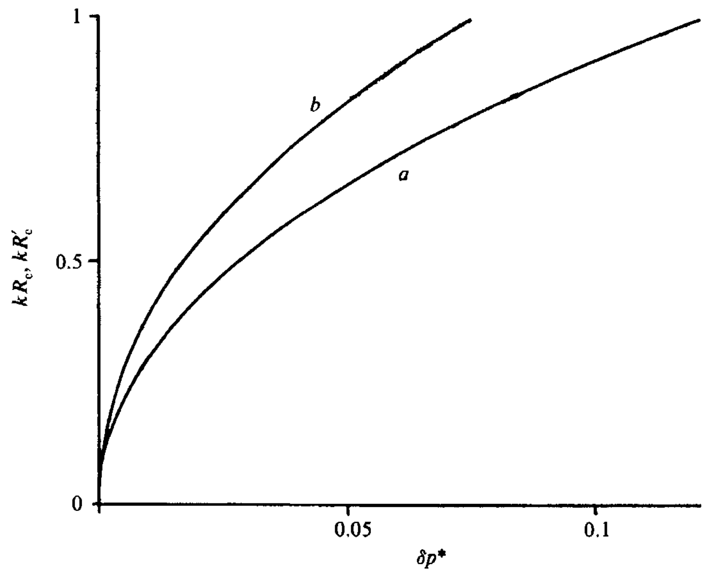

FIGURE 4. The dependence of the cluster radius on the differential pressure δp^*. Curve a corresponds to (27) in which wave propagation beyond the nucleation zone is ignored and curve b to (28), where such an effect is included.

which for a 20 kHz acoustic field gives a cluster radius $R_c \approx 3.0$ mm. This result is reasonably in accordance with experimental results.

It is evident that the definition of p_n from $c = \frac{1}{2}c_0$ influences the calculated value of R_c, but as apparent from figure 4 a different choice satisfying the argument used for defining p_n would not greatly change the result.

In general the nucleation front still moves at high supersonic speed at such small values of kR_c as found above (figure 3). Therefore, after cavities of critical size have developed at $r = R_c$ at time $t = t_n(R_c)$ nucleation is continued outside this surface because relaxation waves from cavities which have passed the critical size move outwards at only sonic speed. At $r = R_c$ the expansion wave of the convergent sound field is reflected as a compression wave and subsequent nucleation events at $r > R_c$ also cause such reflection. Thus, a region of relaxation is formed outside the cavitation region. It spreads at supersonic speed until $r = R_n$, where $v_n = c_0$, and here the relaxation wave field catches up with the nucleation front at time $t = t_n(R_n)$ and excludes further cavity nucleation. For $r > R_c$ the cavitation nuclei again collapse into micro-gas bubbles due to the relaxation waves and the associated reduction of tensile stress so that the medium becomes a single-phase liquid. Only for $r \leqslant R_c$ do the cavities reach and exceed the critical size. This domain now constitutes a genuine cavitating kernel – a cavity cluster – embedded in the single-phase liquid. The pressure in the liquid phase of the cluster is lower than the vapour pressure and the cluster is in an unsteady state, its development being governed by the pressure and velocity field in the surrounding liquid.

3. The growth of the cluster

In the liquid surrounding a cavity cluster which has just developed, the basic pressure is the far field pressure p_0 and the local instantaneous pressure is obtained by superposing the convergent acoustic wave field and the wave field due to its reflection at the boundary of the cavitation region. Inside the cavitation region the pressure can be considered close to (unstable) equilibrium and is connected to the cavity size by (14) as the inertial terms of the bubble equation are small. In general, after relaxation from the critical pressure, the cluster pressure is not much lower than the vapour pressure. The two regions interact at the cluster/liquid interface where the far field pressure operates to collapse the cluster, but at first this effect is surpassed by the acoustic field causing cluster development, later both contribute to its collapse.

After cluster inception by growth of the cavitation nuclei into cavities of critical size, continued growth leads to stress relaxation in the liquid phase of the cluster and new cavities cannot develop. As a consequence, the continued development of the cluster takes place by growth of existing supercritical cavities, and increase of its linear dimension beyond the radius $R = R_c$ is due solely to displacement of the cavities forming the cluster boundary, thus reflecting the increase of the void fraction in the cluster.

The cluster is essentially shielded from the outside pressure field during the time of cavity nucleation, i.e. as long as $v_n > c_0$ (26), and until the convergent acoustic wave has again advanced from $r = R_n$ to R_c at the time $t_g = t_n(R_n) + (R_n - R_c)/c_0$. In this interval of time (see figure 3), the mean void fraction in the cluster grows owing to relaxation spreading from the cluster boundary into the nucleation region $R_c < r < R_n$ and the pressure $p - p'_v \to 0$ (through negative values) as the cavities grow. After time t_g the convergent acoustic wave and the wave reflected from the cluster boundary directly govern the cluster development in conjunction with the far field pressure p_0.

From (12) and (16) we find that when a gas bubble grows to its critical size $\omega_B \to 0$, and beyond this limit a resonance frequency does not exist for isothermal cavities because restoring forces are absent. A tensile stress wave reaching the cavities causes cavity growth due to liquid straining, and simultaneously the equilibrium pressure in the liquid phase rises to a level *higher* than in the undisturbed state as from each cavity exposed to the tensile stress wave a relaxation wave spreads into the liquid at the speed c_0. At the boundary of a cluster of cavities in which the inter-cavity distance is l the tensile stress due to an incident acoustic wave of angular frequency ω is almost immediately relaxed if $c_0/\omega \gg l$. The pressure increase due to cavity growth as derived from (14) is small and the cluster boundary can be considered a totally compliant interface at which the tensile stress wave is reflected with a phase shift π. We might say that the acoustic impedance at the cluster boundary is zero, but as a sound speed c inside the cluster is not defined at vaporous cavitation, its acoustic impedance is not defined just from the properties of the medium. This becomes evident in connection with compression waves.

As at tensile stress the cluster surface constitutes a compliant boundary to the single-phase liquid, the boundary condition leading to (6) does not apply any more. It is now demanded that at $r = R_c$ the pressure perturbation due to the wave field $(\Delta p)_{R_c} = 0$, and the convergent wave (9) is therefore reflected to give

$$\Delta p_f = \left(\rho_0 \frac{\partial \phi}{\partial t}\right)_f = \rho_0 \frac{C}{r} \omega \sin(\omega t - kr + 2kR_c), \tag{29}$$

so that with (5), (9) and (29)

$$\Delta p = \rho_0 \frac{\partial}{\partial t}(\phi_f + \phi_g) = \frac{1}{r}\left(\frac{2\rho_0 W c_0}{\pi}\right)^{\frac{1}{2}} \cos(\omega t + kR_c) \sin(kR_c - kr). \tag{30}$$

Equation (30) gives the velocity perturbation Δu and the displacement ξ of the fluid elements

$$\Delta u = -\frac{\partial}{\partial r}(\phi_f + \phi_g) = \frac{\partial \xi}{\partial t}, \tag{31}$$

which yields

$$\xi = -\left(\frac{2W}{\pi \rho_0 c_0}\right)^{\frac{1}{2}} \cos(\omega t + kR_c) \frac{\cos(kr - kR_c) - \sin(kr - kR_c)\, c_0/(r\omega)}{r\omega}. \tag{32}$$

Equations (29), (30) and (32) are valid for $t > t_g$, $R_c < r < R_c + (t - t_g)\, c_0$ (region IV in figure 3).

From the onset of cavity nucleation until growth driven by the sound field begins at $t = t_g$ only a small void fraction is developed in the cavity cluster and it is in general not decisive. Therefore it may be neglected in a first approach. For $t > t_g$ the standing wave (region IV in figure 3) causes growth of the void volume by

$$\Delta V_a = 4\pi R_c^2 (\xi(t) - \xi(t_g))_{r=R_c} = 4\pi R_c^2\, \Delta\xi(t). \tag{33}$$

However, this increase is counteracted by the far field pressure p_0, which initiates a collapse of the cluster simultaneously with the acoustic field initiating growth of the void volume. During the first period of time the latter effect is the strongest, and, consequently, the cluster boundary remains a material surface, i.e. the fluid elements inside the cluster boundary remain the same. In this period of time the effect of the far field pressure p_0 is essentially described by the Rayleigh–Plesset equation for a single cavity of radius as the cluster

$$R\ddot{R} + \tfrac{3}{2}\dot{R}^2 = -(p_0 - p)/\rho_0, \tag{34}$$

even though the cluster is a bubbly medium. This equation gives the change of cluster radius $\Delta R(t) = R - R_c$ due to the far field pressure, and a resulting total void volume in the cluster

$$\Delta V = 4\pi R_c^2 (\Delta\xi(t) + \Delta R(t)), \tag{35}$$

is obtained.

Accordingly the mean void fraction in the cluster is

$$\bar{\beta}_1 = 3(\Delta\xi(t) + \Delta R(t))/R_c. \tag{36}$$

During the growth period the two terms in (35) do not influence each other to any significant extent, because the pressure at the cluster boundary can be considered almost constant and close to the vapour pressure.

For the cluster of radius $R_c = 3.0$ mm calculated above for a 20 kHz acoustic field we assume that $W = 50$ W, $c_0 = 1483$ m/s, $\rho = 998$ kg/m^3 and with $p_0 = 100$ kPa, $p_n - p_v' = -31.8$ kPa (cavitation number $A \approx 4.4$) equation (24) gives, $t_n(R_c) \approx t_n(0) = -10.66$ µs and for $v_n/c_0 = 1$ equations (24) and (26) give $R_n = 26$ mm and $t_n(R_n) = -7.0$ µs which implies $t_g = 9.0$ µs. With these data (36) gives the development of the mean void fraction $\bar{\beta}_1$ shown in figure 5. The individual contributions from the acoustic field and the Rayleigh–Plesset collapse are also shown. It appears that the maximum value of $\bar{\beta}_1 \approx 0.013$, which for $l = 300$ µm corresponds to a maximum cavity radius $a = 39$ µm, occurs 32.7 µs after cavity nucleation at the focal point. Thus, more than half the period of oscillation of the acoustic field is used for cluster formation and growth.

It should be noted that if the nuclei distribution outside the cluster region differs from that inside the cluster region then the above calculation of t_g is strongly

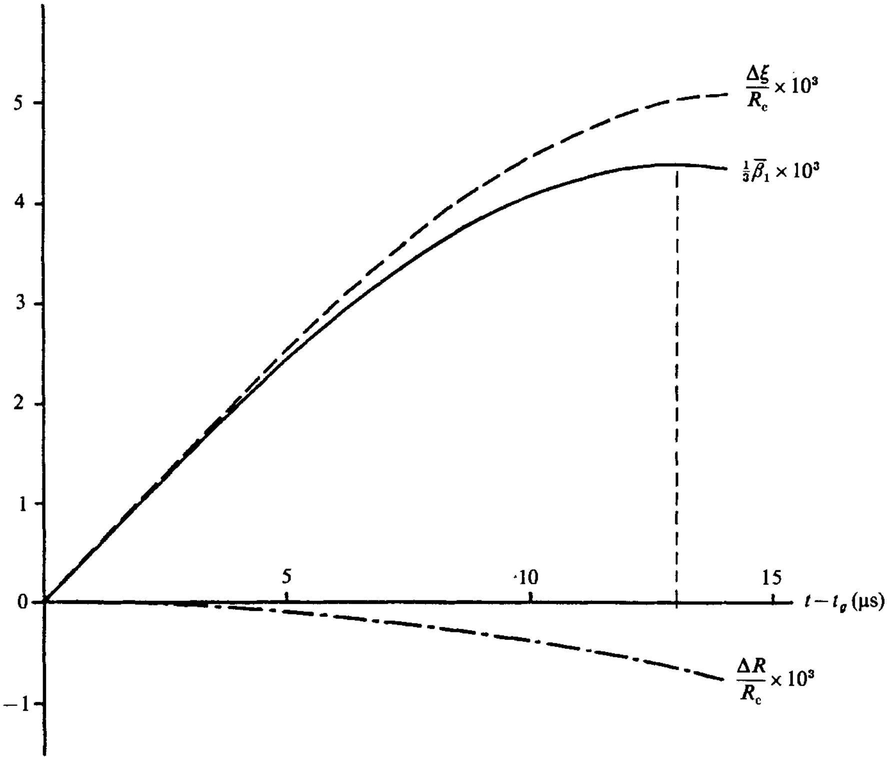

FIGURE 5. The development of the mean void fraction $\bar{\beta}_1$ in the cluster *vs*. time t for the example considered in the text. Further, the cluster boundary displacements $\Delta\xi(t)$ due to the sound field, and $\Delta R(t)$ due to the far field pressure p_0 are given.

affected. If the nuclei generated by successive cluster collapses were not at all spread outside the cluster region we would obtain $t_g = t_n(R_c)$, but such a situation is unlikely to occur.

4. The collapse of the cluster

The maximum void volume in the cavity cluster is reached when growth (33) due to the tensile stress wave is balanced by the Rayleigh–Plesset collapse governed by (34). Now the actual cluster collapse is initiated and (34) ceases to be valid because during collapse the cluster boundary is not a material boundary. It is characteristic that only those cavities close to the instantaneous cluster boundary contribute to the collapse, and so the cluster radius R shrinks by time as cavities at its boundary shrink into gas nuclei, and liquid of the cluster region shifts into liquid of the single-phase region.

The collective effects of multi-bubble collapse were first analysed by Wijngaarden (1964, 1966), who solved the complete system of equations for the collapse of a planar gas bubble region when exposed to high pressure. Later, the collapse of spherical and cylindrical cluster configurations were considered (Hansson & Mørch 1980, Mørch 1980, 1982) by application of Campbell & Pitcher's (1958) shock-wave approximation to the transition from the two-phase to the single-phase condition. Though strictly it is applicable only to gas bubbles, the results are valid for vaporous cavities too. The cluster form is of major importance owing to the energy focusing produced by a convex cluster shape.

If a cavity cluster of void fraction β_1 is embedded in a single-phase liquid and the pressure in the liquid is suddenly raised from the cluster equilibrium value p_1 (equation (14)) to a high level p_0 it starts collapsing. The potential energy of this system, which is connected to the void volume of the cluster, is gradually changed, partly into kinetic energy of the surrounding liquid, and partly lost by dissipation or radiation. This gives an energy balance for the system

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[(p_0-p_v')\,\beta_1\,\Omega_1+\int_{\Omega_2}\tfrac{1}{2}\rho_0\,u^2\,\mathrm{d}\Omega_2\right]=(1-\gamma)\,\tfrac{1}{2}(P_2-p_v')\,\beta_1\frac{\mathrm{d}\Omega_1}{\mathrm{d}t}, \tag{37}$$

which contains the three above-mentioned energy terms, and where Ω_1 and Ω_2 are the volumes of the cluster and of the surrounding liquid, respectively, u is the velocity of the liquid set up by the collapse of the cavities, P_2 is the pressure in the liquid at the cluster boundary, while γ is an energy conservation factor $(0<\gamma<0.5)$. At given cluster configuration, (37) leads to an equation for its collapse.

The undisturbed interior of the cavitation zone is separated from the single-phase liquid by a collapse wave across which the velocity and the pressure grow as the cavities collapse. For a one-dimensional wave the equation of mass conservation

$$\frac{\partial\rho}{\partial t}+\frac{\partial(\rho u)}{\partial x}=0 \tag{38}$$

gives with (17), when $\rho'\approx 0$ and $(\partial/\partial t)+(V_s\,\partial/\partial x)=0$,

$$\frac{\mathrm{d}u}{u-V_s}=\frac{\mathrm{d}\beta}{1-\beta}, \tag{39}$$

where V_s is the speed of propagation of the collapse wave separating the cluster and the single-phase liquid. By integration across the wave we find

$$u=V_s(\beta_1-\beta)/(1-\beta), \tag{40}$$

β being the local void fraction inside the collapse wave. Behind the wave, where $\beta=\beta_2\ll\beta_1$ the velocity of the liquid

$$u_2=V_s\frac{\beta_1-\beta_2}{1-\beta_2}\approx\beta_1\,V_s. \tag{41}$$

Likewise, from the equation of motion for an inviscid fluid

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}=-\frac{1}{\rho}\frac{\partial p}{\partial x}, \tag{42}$$

we obtain with (40) that

$$\mathrm{d}p=-\rho_0\left(\frac{V_s(1-\beta_1)}{1-\beta}\right)^2\mathrm{d}\beta. \tag{43}$$

Integration across the wave gives the local pressure p in the wave

$$p-p_1=V_s^2\,\rho_0\frac{(\beta_1-\beta)(1-\beta_1)}{1-\beta}. \tag{44}$$

With equilibrium conditions behind the wave a driving pressure $p = P_2$ gives the wave speed

$$V_s = \left[\frac{(P_2-p_1)(1-\beta_2)}{\rho_0(\beta_1-\beta_2)(1-\beta_1)}\right]^{\frac{1}{2}} \approx \left[\frac{P_2-p'_v}{\rho_0(1-\beta_1)\beta_1}\right]^{\frac{1}{2}} \tag{45}$$

as at vaporous cavitation $\beta_1 \gg \beta_2 \approx 0$ and $p_1 \approx p'_v$ for cavities significantly larger than the critical size, (15). However, (45) also applies to gas bubbles which are compressed isothermally if only surface tension forces are small, thus confirming earlier results (Mørch 1980).

In planar theory the thickness of the collapse wave is not important, but the application of this theory to the collapse of a non-planar cluster, demands that the wave thickness be small compared with the radius of curvature of the cluster boundary. Therefore, the wave profile, resulting from the collapse course of the individual cavities, is important. Though the cavities do not collapse spherically such collapse is a reasonable approximation, and it allows the Rayleigh–Plesset equation to be used for calculating the cavity radius a *vs.* time t when (44) is used to give the driving pressure. This relation is transformed into one of radius a *vs.* position x in the wave by using $\partial/\partial t + V_s\,\partial/\partial t = 0$, figure 6(*a*).

The associated pressure distribution in the wave is shown in figure 6(*b*) (Mørch 1987*b*). This result essentially agrees with Wijngaarden (1970) for gas bubbles, though these exhibit oscillations in the collapsed state. It is seen that a distance

$$\delta_{0.01} = 2.2\frac{a_1}{(\beta_1(1-\beta_1))^{\frac{1}{2}}}, \tag{46}$$

ahead of the cluster/liquid interface the cavity radius is reduced by only 1% (a_1 is the cavity radius in the undisturbed cluster).

For $\beta_1 = 0.02$ we obtain $\delta_{0.01} \approx 16a_1$ and by (19), $\delta_{0.01} \approx 2.4l$, which shows that essentially only the outermost cavities participate in the collapse. It means that for spherical cavity clusters of initial radius $R = R_0 \gg l$, one-dimensional collapse theory is applicable until only a small number of cavities remain and $R \approx 2l$.

For a spherical cavity cluster of radius R, inside which $u = 0$, $\beta_1 =$ constant, while in the surrounding liquid

$$u = u_2\left(\frac{R}{r}\right)^2 = \beta_1\dot{R}\left(\frac{R}{r}\right)^2 \tag{47}$$

obtained by (41), we find with (37) using $\mathrm{d}\Omega_1 = 4\pi R^2\,\mathrm{d}R$ and $\mathrm{d}\Omega_2 = 4\pi r^2\,\mathrm{d}r$, $r\in[R;\infty]$ that

$$R\ddot{R} + (\tfrac{3}{2}-\tfrac{1}{2}(1-\gamma)(1-\beta_1))\dot{R}^2 = -\frac{p_0-p'_v}{\rho_0\beta_1}. \tag{48}$$

For $\beta_1 = 1$, (48) reduces to the usual Rayleigh–Plesset equation for a single cavity, whereas a cluster of small β_1 collapses essentially as a single cavity exposed to a far field pressure enhanced by the factor $1/\beta_1$. The collapse is shown in figure 7(*a*) for $\beta_1 \ll 1$ with $\gamma = 0 \wedge 0.5$, and the corresponding pressure developments at the cluster boundary P_2 calculated from (45) are shown in figure 7(*b*). It is noticed that P_2 rises sharply as the cluster collapses, and the last cavities collapse at a very high pressure. However, as a consequence of the non-vanishing thickness of the collapse wave, P_2 actually remains finite when $R \to 0$. During collapse of each cavity a radial flow towards its centre is set up. If viscous and thermal losses are neglected and the collapse is spherical, the associated kinetic energy is converted and radiated as a

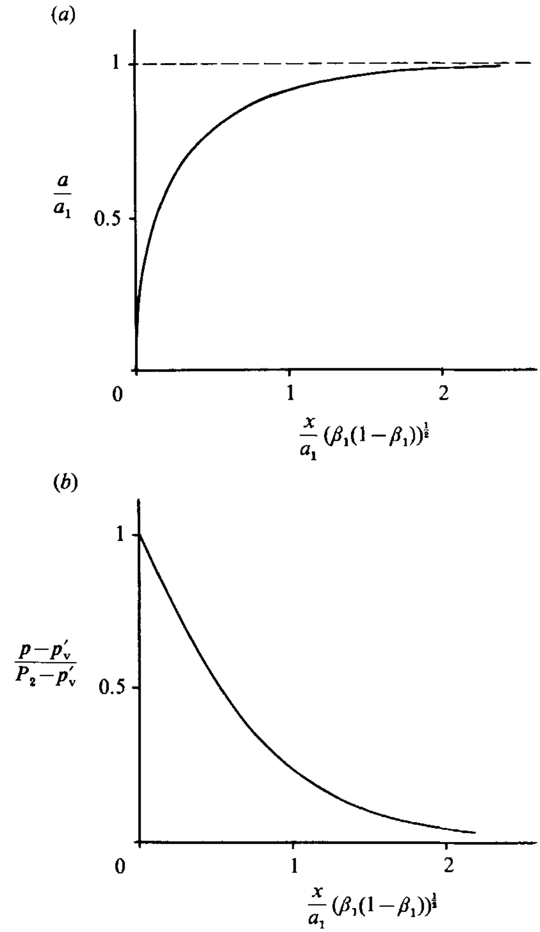

FIGURE 6. (*a*) The radius a of a cavity *vs*. position x in the collapse wave. (*b*) The pressure distribution in the collapse wave (Mørch 1987*b*).

spherical acoustic wave. Half the energy is radiated away from the cluster and is lost, but the other half is radiated into the cluster and contributes to its collapse, so that this energy is conserved in the system, i.e. in (37), $\gamma = 0.5$. In the case of a fully non-spherical cavity collapse in which all the radial energy is converted into kinetic energy of the collapse jet, this energy is essentially lost by dissipation during jet penetration into the liquid. In this case $\gamma = 0$. Actually, a shock wave is always radiated from each cavity at collapse, but thermal and viscous losses are not negligible, and the cavity collapses are not spherical. Therefore, $0 < \gamma < 0.5$.

In the dissipation term given as the right-hand side of (37) $\frac{1}{2}\beta_1(P_2 - p'_v)$ represents half the potential energy per unit of volume of the cluster at the instantaneous cluster boundary pressure P_2. The other half is inherently converted into kinetic energy of the liquid elements which shift from the cluster region to the single-phase region, their velocity changing from $u = 0$ to $u = u_2$ at collapse of the cavities.

If we superpose an acoustic field on the pressure and velocity fields of the collapse driven by the far field pressure p_0 they jointly have to satisfy the cluster boundary conditions of fluid velocity $\beta_1 \dot{R}$ and pressure $\rho_0 \beta_1 (1 - \beta_1) \dot{R}^2$ given by (41) and (45).

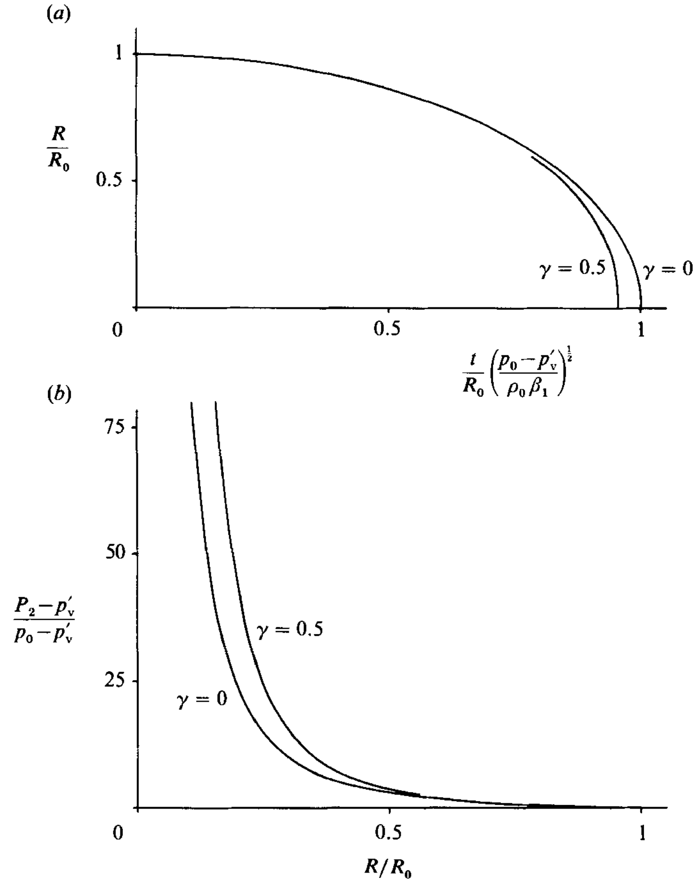

FIGURE 7. (*a*) The cluster radius R *vs.* time t during the collapse of a spherical cavity cluster of initial radius R_0 for $\beta_1 \ll 1$, $\gamma = 0 \wedge 0.5$. (*b*) The corresponding pressure P_2 at the cluster boundary *vs.* the cluster radius R.

Therefore, at spherical cluster collapse the equation for the velocity of the liquid (47) is modified to be

$$u = (\beta_1 \dot{R} - \Delta u(R))(R/r)^2, \tag{49}$$

and the pressure driving the collapse (37) is

$$P_2 - p'_v = \rho_0 \beta_1 (1-\beta_1) \dot{R}^2 - \Delta p(R), \tag{50}$$

in which $\Delta u(R)$ and $\Delta p(R)$ depend on the matching conditions of the convergent acoustic wave to the cluster collapse wave.

We notice from the above discussion that the wave speed V_s adjusts itself to give a collapse wave pressure P_2 and a velocity distribution in the wave which match at its rear to the external single-phase flow. From (41) and (45) we find

$$\frac{dP_2}{du_2} = 2\rho_0 \frac{1-\beta_1}{\beta_1} u_2 \approx 2\rho_0 (1-\beta_1) V_s, \tag{51}$$

showing that though a sound speed is not defined inside the cluster a wave dependent acoustic impedance can be ascribed to the cluster. This dynamic impedance is smaller

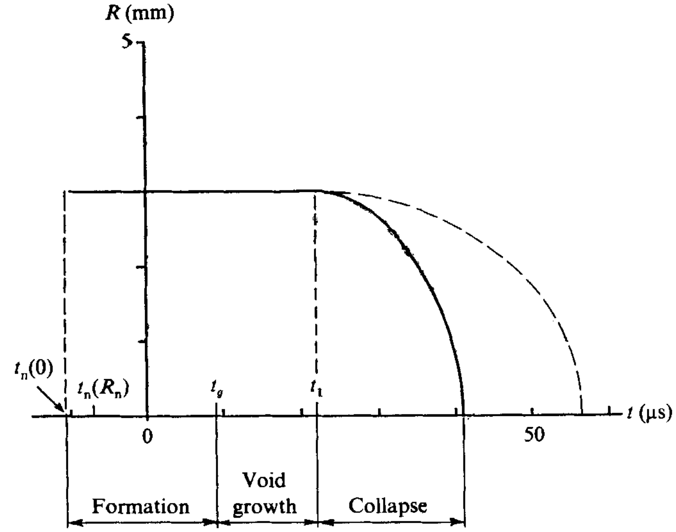

FIGURE 8. The radius R of the cavity cluster *vs.* time t for the example discussed in the text. ——, collapse calculated from (53); ----, collapse calculated for the case when the sound field is neglected.

than the acoustic impedance of the liquid phase itself. Further, $\mathrm{d}u_2$ is phase locked to $\mathrm{d}P_2$.

A planar compressional acoustic wave being superposed on a planar collapse wave is therefore partially reflected as a tensile stress wave. However, in the near field $kr \ll 1$ of a spherically symmetrical system, there is a $\frac{1}{2}\pi$ phase displacement between Δp_g and Δu_g, (4) and (9), of the convergent wave (3). As the collapse wave cannot respond to these perturbations the wave is reflected at $r = 0$ as if the cluster were not there, and the acoustic field is given by (8) and (11) which with $kr \ll 1$ leads to

$$\Delta u(R) \approx 0,$$

and

$$\Delta p(R) = -\omega\left(\frac{2\rho_0 W}{\pi c_0}\right)^{\frac{1}{2}}\frac{\sin kR}{kR}\cos\omega t \quad (kR \ll 1). \tag{52}$$

This is equivalent to a total reflection without phase shift at the cluster boundary.

We now apply (52) to (49) and (50). The pressure increase at the cluster boundary due to the sound field is superposed on the effects of the far field pressure p_0 and is balanced by the transient collapse, and from (37) we obtain

$$R\ddot{R} + (\tfrac{3}{2} - \tfrac{1}{2}(1-\gamma)(1-\beta))\dot{R}^2 = -\frac{p_0 - p'_v + (1-\gamma)\frac{1}{2}\Delta p(R)}{\rho_0 \beta_1}. \tag{53}$$

Thus, the effect of the sound field on the cluster collapse is equivalent to an increase of the far field pressure by $\frac{1}{2}(1-\gamma)\Delta p(R)$, in which $\Delta p(R)$ is obtained from (11). For the numerical example considered in §§2 and 3, (53) gives a collapse time of 19 μs (figure 8). With 32.7 μs for cluster inception and growth the complete cycle demands slightly more than one period of oscillation of the 20 kHz acoustic field. The numerical results seem quite satisfactory, but it should be borne in mind that the data used for the calculation, especially concerning the cavitation nuclei and their distribution, are very approximative. Therefore, a detailed quantitative evaluation of the theoretical model is not justified.

5. Discussion

This paper discusses why a cluster of small cavities is formed in an acoustic cavitation process instead of just a single large cavity, how the cluster develops and how it collapses at the joint exposure to the acoustic field and the far field pressure. A cavitation number for the acoustic field is obtained. Further, it is found that a cavity cluster exhibits a dynamic acoustic impedance, which is connected to the bubble dynamics. Data for quantitative calculation of a cavitation cycle are chosen from the experimental evidence available in the literature. These calculations lead to plausible results concerning the cluster size, the maximum void fraction in the cluster and the duration of different parts of the cluster life cycle. However, experimental work carried out at controlled conditions, especially with respect to the cavitation nuclei and their distribution, is lacking. The few series of high-speed photographs of cavity cluster life cycles that exist are sparsely supplied with information concerning the acoustic field as well as the liquid.

It is supposed in the present paper that the number density and size of the gaseous cavitation nuclei reach an equilibrium during continuous operation of the acoustic field. The first stress wave may cause a single solid or gaseous cavitation nucleus near the focal point to grow into a large cavity into which gas diffuses during its lifetime. At collapse it may split into two or more viable gas nuclei that grow in the next cycle to smaller maximum sizes than the first one, and so the gas diffusion into each of them is reduced. Accordingly, the tendency to splitting into an increased number of viable nuclei decreases when the number of nuclei increases as a result of successive stress waves. Therefore, at given gas content and temperature of the liquid a continuous acoustic field is expected to lead to a balance of the number density of cavitation nuclei. Due to acoustic streaming and to spatial instability of the cluster collapse process it is conjectured that the nuclei generated in the cavitation process spread into the surrounding liquid, and survive so that an approximately uniform dynamic tensile strength of the liquid can be assumed.

A better knowledge of the above conditions of cluster formation is primary to more reliable calculations of the cluster life cycle, but also the approximations made in this paper concerning the conditions of wave reflection at the cluster boundary, particularly during cluster growth, and the transition to the collapse phase, should be considered in further depth.

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